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Rigor and Mathematics in Indiana Schools

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The phrases "mathematical rigor" and "rigorous mathematics" have bounced around from Indianapolis to school corporations throughout the state ever since the Common Core State Standards were introduced in 2010 and then subsequently replaced by the Indiana Academic Standards for Mathematics in the spring of 2014. The Indiana Department of Education website, local newspapers, Indiana business websites, and school district websites have offered numerous comments. A sampling of those headlines and statements include the following:

- The Indiana Academic Standards for Mathematics are the "result of a process designed to identify, evaluate, synthesize, and create the most high-quality, rigorous standards for Indiana students." (Indiana Academic Standards, 2014)
- The 2014 Mathematics Standards draw on strengths from Indiana's 2009 standards, the ADP Benchmarks, and the Common Core State Standards. Aspects of all of these documents are incorporated in the 2014 Mathematics Standards, resulting in standards that are generally rigorous, coherent, focused, specific, clear and accessible, and measurable.
- As a hallmark of their rigor, they provide an appropriate balance between conceptual understanding, procedural fluency and application to problem solving. (A Review of the Draft 2014 Indiana K-12 Content Standards for College and Career Readiness in English Language Arts and Mathematics. Submitted by Achieve at the request of Governor Pence, April 1, 2014).
- Indiana withdrawal of Common Core standards causes changes. "I believe students will (graduate) with the same body of knowledge that's always existed in language arts and mathematics. Two plus two will always equal four. But they're going to be able to apply those skills to a higher level of rigor," said Danielle Shockley, assistant superintendent for the Indiana Department of Education. (Jessica R. Key, Indianapolis Recorder, Thursday, April 17, 2014, 1:30 pm).

Strangely, although not uncommon in mathematics documents of other states, the Indiana Department of Education document does not provide readers with a workable, action-led discussion of the phrases *mathematical rigor* and *rigorous mathematics*. What do these phrases mean when we consider the mathematics education of Indiana students. What are the expectations for Indiana mathematics teachers at the elementary, middle school, or high school levels as they implement the Indiana Academic Standards and also strive to provide a *rigorous* curriculum? What does a mathematics classroom look like when mathematical rigor is being implemented? Are we already doing some of the necessary actions? How can we provide a rigorous K-12 mathematics curriculum if we do not know what *rigor* or *rigorous* mean?

Answering these and other related questions requires a definition of the term *mathematical rigor*—one that provides an opening, an avenue, or a focus on which teachers, leaders, and administrators can get a manageable and workable handle on rigor. A definition, however, is only a start. Implementing rigor is a big change for most teachers; educators know that making such a change is difficult because instruction and student learning must, in most cases, drastically change.

Definition of Rigor

Articles, various state curricula, and numerous online presentations discuss mathematical rigor and a rigorous mathematical curriculum. Subheadings within these often suggest that a definition of rigor will be forthcoming, yet one never emerges from the discussions. Instead, the discussions often suggest what mathematical rigor is *not*: a measure of the quantity of material covered or the number of times it is covered.

In researching the subject of rigor, we have located several findings that provide a basis for our definition of mathematical rigor. They come from publications of the National Research Council over a period of several years:

- Students are talking and sharing (NRC, 2004).
- Students are collaborating in small groups (NRC, 2005).
- Students are thinking and reasoning (NRC, 1999).
- Students are explaining their thinking (NRC, 2001).
- Students are using manipulatives to construct meaning (NRC, 2000).
- Students are devoting appropriate time to solving challenging problems (NRC, 2000).

Continued...

About the Journal

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These statements suggest that both students and teachers must be actively engaged in the classroom in order for students to learn mathematics with rigor. Therefore, we offer the following two-part definition of rigor.

Mathematical Rigor

Mathematical rigor is the depth of interconnecting concepts and the breadth of supporting skills students are expected to know and understand.	*Mathematical rigor is the effective, ongoing interaction between instruction and student reasoning and thinking about concepts, skills, and challenging tasks that results in a conscious, connected, and transferable body of valuable knowledge for every student.*
Content	Instruction

Students' thinking and reasoning skills are increased through active engagement. Classroom discourse should become a routine, as it greatly enhances and clarifies such reasoning and thinking. From an equity standpoint, no student should be allowed to passively disengage from the mathematical learning and teaching that is taking place.

Lesson Initiators That Lead to Rigor

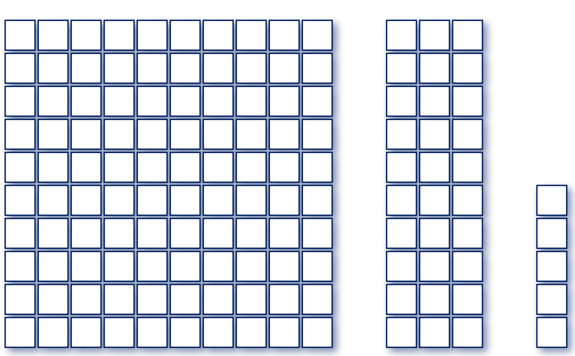
Due to space limitations, we offer two samples of what we consider "openings" for mathematical rigor to occur. First, consider a traditional setting in Grade 2 where students are learning place value with three-digit numbers—a standard within the Number Sense strand:

> 2.NS.7: Use place value understanding to compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.

The introduction to the Grade 2 standards (p. 3) notes, "*The Mathematics standards for grade 2 are supplemented by the Process Standards for Mathematics*. For our particular setting, *Process Standard 4. Model with mathematics* and *Process Standard 5. Use appropriate tools strategically* are important.

Using Base-10 blocks, students might encounter a question such as the following in their textbooks:

> What three-digit number does the following arrangement of Base-10 blocks represent?



The answer is 135! Is there any student collaboration required to answer this question? Is there any engagement involved? Is rigor present? In most cases, the answer to these questions is a resounding "No!" To add rigor to the lesson, we add the following directions:

> There are many three-digit numbers that can be made using any combination of the Base-10 blocks shown. How many can you find? Work with a partner and use your blocks to find the three-digit numbers.

The restated problem now provides multiple entry points for *all* students.

Some partner pairs will immediately begin to list three-digit numbers on their papers. Other pairs will take Base-10 blocks and move them into various configurations.

A variety of questions then begin to surface. Do we have to use all of the blocks for a number? Can there be a 0 in the hundreds place? What about a 0 in the ones place and tens place?

As students share their three-digit numbers, they notice patterns:

- To have a three-digit number, there must be a 10-by-10 flat in the hundreds place.
- There are actually four possibilities for the tens place. 0, 1, 2, or 3.
- There are actually six possibilities for the ones place: 0, 1, 2, 3, 4, or 5.

Providing time for the discussion of a challenging problem is an important aspect of mathematical rigor. Such student-student or student-teacher discussion calls on an additional Process Standard that involves "*constructing viable arguments and critiquing the reasoning of others"* as students agree that there are 24 possible three-digit numbers. For higher grades, the result leads to a multiplication problem: 1 x 4 x 6 = 24 possible numbers. Middle school and high school mathematics teachers recognize this result as an application of the Fundamental Counting Principle.

130	120	110	100
131	121	111	101
132	122	112	102
133	123	113	103
134	124	114	104
135	125	115	105

The inclusion of engagement, collaboration, and use of manipulatives adds rigor to a traditional lesson on place value. Skills are necessary; reasoning and thinking are evident. The problem also provides substantial mathematics for further discussion of concepts in the Finite and Probability and Statistics strands.

A second example of an "opening" for mathematical rigor to occur comes from the secondary Algebra I curriculum in the Systems of Equations and Inequalities strand. Grade 8 does include one standard in its Algebra and Functions strand dealing with systems of equations (8.AF.8); however, it is limited in its scope.

In the Algebra I Systems of Equations and Inequalities strand, there are four standards. Our sample focuses only on the first:

AI.SEI.1: Understand the relationship between a solution of a pair of linear equations in two variables and the graphs of the corresponding lines. Solve pairs of linear equations in two variables by graphing; approximate solutions when the coordinates of the solution are non-integer numbers.

Typically, students are solving and graphing systems of equations as suggested by the standard. In doing so, they learn a series of related concepts. The graphs of a system of two linear equations might:

- Intersect at a point and therefore have one solution.
- Be parallel and therefore have no solutions.
- Represent the same line and therefore have an infinite number of solutions.

If they are parallel, the slopes of the two lines are the same and the y-intercepts are different. If they represent the same line, then one equation is a multiple of the other with the same slope and y-intercept. If they intersect, then the lines have different slopes.

Consider the following traditional problem; although such a problem is necessary for teachers to scaffold mathematics instruction, it does not provide for rigor:

For which system of two linear equations are the graphs parallel?

For which system of two linear equations are the graphs intersecting?

$2x + 3y = 5$	$2x + 4y = 3$	$2x + 6y = 5$
$2x + 3y = 8$	$4x + 8y = 6$	$2x + 7y = 8$

We contend that the following type of problem offers mathematical rigor:

Use all of your number tiles, 0–9, to create three systems of equations so that one system has exactly one solution, a second system has no solution, and the third system has an infinite number of solutions.

$2x + \_\_\_ y = \_\_\_$	$\_\_\_ x + 4y = \_\_\_$	$\_\_\_ x + 6y = \_\_\_$
$\_\_\_ x + 3y = 0$	$4x + \_\_\_ y = 1$	$\_\_\_ x + 7y = \_\_\_$

Referring to our two-part definition, mathematical rigor requires students to interconnect concepts, reason, think, and communicate about mathematical language such as, in this case, the concepts involving slopes, y-intercepts, parallel lines, intersecting lines, same line, one solution, no solution, and infinite number of solutions. All of these ideas are part of *Process Standard 6. Attend to precision.* At the same time, *Process Standard 7. Look for and make use of structure* can be considered using patterns as suggested above. This particular type of example with systems of two linear equations also lends itself to several other Process Standards: *Reason abstractly and quantitatively, Construct viable arguments and critique the reasoning of others*, and *Use appropriate tools strategically*.

Ongoing Efforts to Implement Rigor

The Department of Education document has a subheading for each grade level or course entitled, "What are the Indiana Academic Standards NOT?" Item 2 notes that "the standards are not instructional practices" and goes on to state that "the standards do not define how teachers should teach." Although we agree with such statements, our definition of mathematical rigor involves two parts: content and instruction. We believe that both must be present in our classrooms, regardless of grade level, in order to have a rigorous mathematics program. Student thinking and reasoning, engagement, classroom discourse, collaboration, manipulatives and other tools are major factors in implementing rigor into mathematics classrooms. For all of this to happen, however, district, school, and classroom changes that focus on rigor will require a multi-layered professional development offering—one that focuses on mathematical content, the Indiana Process Standards, and overall strategies that promote rigor in the classroom. Constant, consistent effort is necessary for teachers and classrooms to change, to make rigor a common theme for all grades, K–12.

References

Hull, T., Balka, D., & Harbin-Miles, R. (2014). *Realizing rigor in the mathematics classroom*. Monterey, CA. Corwin Press.

National Research Council. (1999). *Improving student learning: A strategic plan for education research and its utilization*. Washington, DC: National Academy Press.

National Research Council. (2000). *How people learn: Brain, mind, experience, and school.* Washington, DC: National Academy Press.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, D.C: National Academy Press.

National Research Council. (2004). *Engaging schools: Fostering high school students' motivation to learn.* Washington, DC: National Academy Press.

National Research Council. (2005). *How students learn: Hhistory, mathematics, and science in the classroom.* Washington, DC: National Academy Press.

Assessing Basic Fact Fluency

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Have you had it with timed tests, which present a number of concerns and limitations? Try a variety of alternative assessments from this sampling that allows teachers to accurately and appropriately measure children's fact fluency.

Think about how you assess reading fluency. Does your assessment plan involve listening and observing as children read as well as asking reading comprehension questions? Now imagine what you might learn about students' reading fluency if you used *only* timed quizzes. How would your confidence in your assessment change?

Formative assessments—including observations, interviews, performance tasks, and journaling—have become common practice in many classrooms, with a recognition that by using different ways to assess children, we gain a more comprehensive, accurate picture of what they know, what they do not know, and their misconceptions. These data are then used to design instruction accordingly (Wiliam 2011). Yet, in spite of this trend in other areas of education, timed, skill-based assessments continue to be the prevalent measure of basic mathematics facts achievement. As a result, many rich opportunities for assessing basic fact fluency are lost. In this article, we share a variety of ways to formatively assess basic fact fluency. We define fluency, raise some issues related to timed testing, and then share a collection of classroom-tested ideas for authentic fact fluency assessment.

Defining fluency

A variety of interpretations exist for what procedural fluency (in general) and basic fact fluency (specifically) mean. Fortunately, recent standards, research, and reports provide a unified vision of what these terms signify. The Common Core State Standards for Mathematics (CCSSM) document describes *procedural fluency* as "skill in carrying out procedures *flexibly*, *accurately*, *efficiently*, and *appropriately*" (CCSSI 2010, p. 6). Likewise, Baroody (2006) describes basic fact fluency as "the efficient, appropriate, and flexible application of single-digit calculation skills and . . . an essential aspect of mathematical proficiency" (p. 22). These definitions reflect what has been described for years in research and standards documents (e.g., NCTM 2000, 2006; NRC 2001) as well as CCSSM grade-level expectations related to basic facts (see **table 1**).

Notice that the CCSSM expectations use two key phrases; the first is to *fluently* add and subtract (or multiply and divide), and the second is to *know from memory* all sums (products) of two one-digit numbers. To assess basic fact *fluency*, all four tenets of fluency (flexibility, appropriate strategy use, efficiency, and accuracy) must be addressed. Additionally, assessments must provide data on which facts students *know from memory*. Timed tests are commonly used to supply such data—but how effective are they in doing so?

Table 1

Past mathematics documents—as well as current standards, studies, and reports—offer a unified vision of what procedural fluency means.

CCSSM standards that address fluency and mastery with basic facts (italics added)	
Kindergarten K.OA.A.5	*Fluently* add and subtract within 5.
Grade 1 1.OA.C.6	Add and subtract within 20, *demonstrating fluency* for addition and subtraction within 10. *Use strategies* such as counting on; making 10; decomposing a number leading to a 10; using the relationship between addition and subtraction; and creating equivalent but easier or known sums.
Grade 2 2.OA.B.2	*Fluently* add and subtract within 20 using *mental strategies* (refer to 1.OA.6). By the end of grade 2, *know from memory* all sums of 2 one-digit numbers.
Grade 3 3.OA.C.7	*Fluently* multiply and divide within 100, using *strategies* such as the relationship between multiplication and division (e.g., knowing that 8 x 5 = 40, one knows 40 / 5 = 8) or properties of operations. By the end of grade 3, *know from memory* all products of 2 one-digit numbers.

Limitations and risks of timed mathematics tests

Timed tests offer little insight about how flexible students are in their use of strategies or even which strategies a student selects. And evidence suggests that efficiency and accuracy may actually be negatively influenced by timed testing. A study of nearly 300 first graders found a negative correlation between timed testing and fact retrieval and number sense (Henry and Brown 2008). Children who were frequently exposed to timed testing demonstrated *lower* progress toward knowing facts from memory than their counterparts who had not experienced as many timed tests. In fact, growing evidence suggests that timed testing has a negative impact on students (Boaler 2012, Henry and Brown 2008, Ramirez et al. 2013). Surprisingly, the anxiety that many children experience over timed testing is unrelated to how well they do on the tests. Even high-achieving children share such concerns as, "I feel nervous. I know my facts, but this just scares me" (Boaler 2012). Math anxiety appears as early as first grade, and this anxiety does not correlate with reading achievement (Ramirez et al. 2013). In other words, children's anxiety is specific to mathematics, not general academic work. And the struggling learner is not the only one who experiences anxiety: Ramirez and his colleagues found that children demonstrating a higher use of "working memory" (i.e., those who tended to use mathematical strategies that were more sophisticated) experienced the most negative impact on achievement as a result of math anxiety. Thus, it appears that some of our best mathematical thinkers are often those most negatively influenced by timed testing.

Fortunately, children can learn facts effectively without the use of timed testing. In a longitudinal study of twenty second graders, Kling found that without any timed testing or other rote fact activities, by the end of the year, the children demonstrated automaticity with addition facts (solved within 3 seconds) 95 percent of the time. Interestingly, the children performed strategies (e.g., making ten or near doubles) so quickly that it was impossible to distinguish between highly efficient strategy application and "knowing from memory." Since the beginning of first grade, fact practice for these children had involved (a) activities within textbook lessons (b) weekly fact games and (c) activities such as Quick Images with ten frames that were used to foster discussion around strategies (see Kling 2011, Bay-Williams and Kling in press). This research suggests that timed assessments and drill may not be necessary for children to achieve fact mastery.

If timed mathematics assessments have questionable value and potentially negative psychological, emotional, and educational impact, why are they still so frequently used? We commonly hear three reasons. First, fluency is interpreted as synonymous with speed. We have already addressed that fluency is more comprehensive than speed. Second, some feel that timed tests prepare children for high-stakes tests. The research shared here convincingly shows it may do the opposite. Third, timed tests are the only assessments widely available for assessing fluency of basic facts. As we seek to rectify this last concern, the remainder of this article shares methods for more comprehensively and appropriately assessing students' basic fact fluency.

Figure 1

Below are protocols for student interviews, which are a way to quickly assess all four categories of fluency and see if a student just knows a fact. Insights from a selection of interviews can inform instruction for the whole class.

Protocol A. *Assess* **fluency**

1. Write 4 × 5 on a card *[point at card]*.
   What does 4 × 5 mean?
2. What is the answer to 4 × 5?
3. How did you find the answer to 4 × 5?
   Could you find it another way?
4. If your friend was having trouble remembering this fact, what strategy might you suggest to him or her?

Protocol B. *Assess* **flexibility** *and* **strategy selection**

1. What is 8 + 5?
2. How can you use 8 + 2 to help you solve 8 + 5?

OR

1. How can you use 3 × 7 to solve 6 × 7?

Protocol C. *Assess* **use of appropriate strategy**
(adapted from Henry and Brown 2008)

Probes

What is 7 + 8?
How did you figure it out? *[Ask regardless of how quickly or accurately they solve the fact.]*

Codes

R = Recall
A = Automatic (within 3 seconds)
M10 = Making 10 Strategy
ND = Near Doubles Strategy
D = Some other derived fact strategy
CO = Counting on
CA = Counting all
MCA = Modeling and counting all

Using formative assessment strategies

With an eye on the aspects of fluency (accuracy, efficiency, flexibility, and appropriate strategy selection), we can use various assessment strategies to see what students know (and do not know) and determine what our next instructional steps might be. All are approaches we have used with children in grades 1–4, and when used in combination with one another, these methods provide a comprehensive picture of a student's level of fact mastery.

1. Interviews

Interviews provide the extraordinary opportunity to hear children explain what they know about a topic in a discussion format, during which teachers can ask follow-up and clarifying questions (Hodges, Rose, and Hicks 2012; Van de Walle et al. 2014). The insights gained from listening to a child can be invaluable for planning individualized instruction or interventions. And insights from a selection of students can inform instruction for the whole class. Consider which aspects of fluency to address using the questions posed in each of the sample interview protocols (see **fig. 1**). We see the interview as a quick way to get at all four categories of fluency (as well as to see if a student just knows the fact). *Accuracy* is assessed as soon as the student responds, and *efficiency* is observed on the basis of how long it takes a student to solve the fact. *Flexibility* and *appropriate strategy selection* are addressed by such follow-up prompts as, "How did you figure it out?" or "How could you use this strategy to solve this fact?" Codes, such as those suggested in Protocol C, can facilitate recording during an interview (see **fig. 1**).

Interviews need not be one-on-one, sit-down events. They can be quick exchanges in the midst of other activities. For example, as students are lining up, ask, "Aaron, what is six times seven? How did you figure it out?" Furthermore, interviews have an added benefit of allowing students the opportunity to self-correct. For example, during interview assessments with thirty-eight beginning first graders, Kling found that 54 percent of the time children self-corrected incorrect answers as they explained how they figured out the fact. Furthermore, the children's self-reported strategies (i.e., "I counted on my fingers" or "I just knew it") were consistent with what the interviewer was able to observe 97 percent of the time. This suggests the potential of interviews as highly reliable and informative assessment tools.

2. Observations

Observation is a natural part of teaching, and recognizing which strategies students know can supply valuable insights to help support students as they learn new strategies and tackle unknown facts. To create organized and accurate records of observations, a list of students and facts can be attached to a clipboard (see **table 2a**); or a list can be tracked on the basis of which strategies students use (see **table 2b**). Equipped with these charts, you can observe as students engage in facts games, such as mathematized versions of classic games of War, Go Fish, Concentration, Old Maid, and Memory (for other games to teach and review basic facts, see Forbringer and Fahsl 2010; Kamii and Anderson 2003; Van de Walle, Karp, and Bay-Williams 2013; Kling 2011; Bay-Williams and Kling, in press; Kling and Bay-Williams, in press).

Continued…

Table 2

Codes can facilitate recording during an interview.

(a) Use an accuracy table to review students' progression with addition facts.

	Within 5	Foundational Facts			Within 10	Within 20
Name/facts		0,1,2	Combinations that make 10	Doubles		
Nicholas						
Kayla						
Cynthia						
Robbie						
...						

(b) A table can show the frequency of addition fact strategy use at a glance.

Name/ strategies	1 more/ 2more	Combinations that make up 10	Making 10	Doubles	Find 5s	Applies Commutativity
Nicholas	+			+	+	+
Kayla	+			+	+	
Cynthia		+		+		+
Robbie	+	+	+			+
...						

A critical aspect of meaningful use of games is to ask students to tell their teammates both the answer *and* how they found it. As students turn over cards, observe to see and hear how *efficient* each student is as well as whether he or she chose an *appropriate strategy* or if they just knew. The teacher might observe, for example, that many students are more *efficient* at solving 5 + 3 than they are at 3 + 8. These students may "just know" facts within ten but may apply strategies for the facts that have sums over ten. Such insights gained through observation can help the teacher select appropriate activities for continued learning and practice.

To enhance opportunities for assessing students during game play, consider having groups rotate through centers, stationing the teacher at one center and using probing questions, such as "How did you figure that out?" or "Are there any other ways you could figure it out?" One first-grade teacher had the following to say after using an observational checklist to formatively assess her students:

> This is an important tool that provides a more comprehensive check of which specific strategies a student has successfully mastered toward developing fluency with their basic facts. CCSSM provide specific strategies that students are expected to understand and use, and the chart provides me the opportunity to learn which strategies are being used effectively and where there are opportunities for further instruction and practice.

3. Journaling

Writing provides an excellent opportunity to assess flexibility and understanding of strategy selection and application. Children at any grade level can find ways to incorporate pictures, words, and numbers to communicate their strategies. For example, **figure 2** shows a variety of first graders' responses to the journal prompt, "If your friend did not know the answer to 4 + 5, how could he figure it out?" Carefully review the responses, considering what they illustrate about the strategies used by the children. In contrast to what can be learned from a child's answer to 4 + 5 on a timed test, these samples offer rich opportunities to recognize which children can appropriately select and explain an efficient strategy for the task. This is important for deepening strategy understanding and also is reflected in the expectations of CCSSM and the related, forthcoming assessments. For example, Smarter Balanced Assessment Consortium (SBAC) lists the following as "evidence required" for grade 3. Note the application of strategies inherent in these expectations. The student—

- multiplies and divides facts *accurately* ;
- multiplies and divides facts *using strategies*, such as the relationship between multiplication and division or properties of operations; and
- uses multiplication and division facts (SBAC 2012) (emphasis added).

Meaningful writing tasks can be used across grade levels and operations. **Table 3** presents a collection of writing prompts that address the four components of fluency. Having an opportunity to write about strategies on a weekly basis engages students in self-reflection and self-monitoring as well as emphasizes the importance of strategies in practicing basic facts.

Figure 2

Various responses to a journal prompt illustrate the strategies that first graders used and reveal which children were able to appropriately select and explain an efficient strategy for the task.

If your friend did not know the answer to 4 + 5, how could he figure it out?

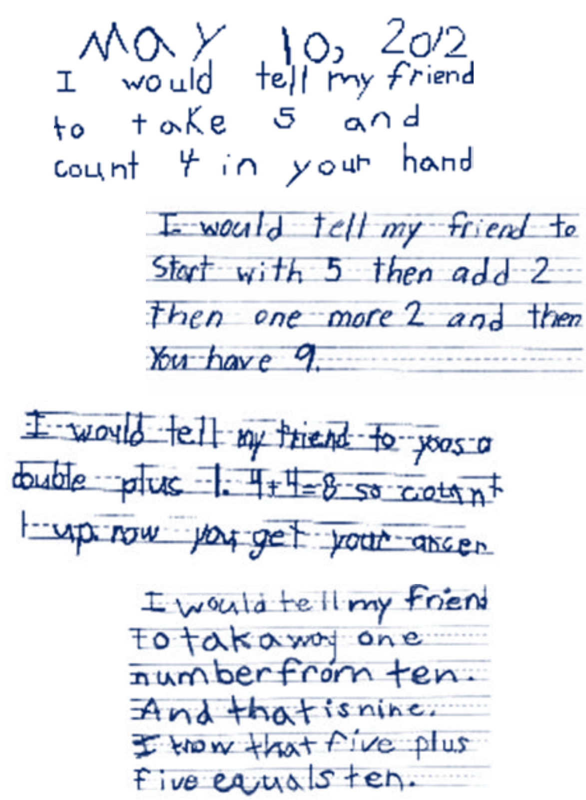

Table 3

This collection of prompts addresses the four components of fluency with basic facts. Writing about their strategies on a weekly basis engages students in self-reflection and monitoring, as well as emphasizes the importance of strategies in practicing basic facts.

Writing prompts for developing fluency with the basic facts

Appropriate strategy selection	Flexibility
• Explain how to use the "count on" strategy for 3 + 9. • What strategy did you use to solve 6 + 8? • A friend is having trouble with some of his times 6 facts. What strategy might you teach him? • Emily solved 6 + 8 by changing it in her mind to 4 + 10. What did she do? Is this a good strategy? Tell why or why not.	• How can you use 7 × 10 to find the answer to 7 × 9? • Solve 6 × 7 using one strategy. Now try solving it using a different strategy. • Emily solved 6 + 8 by changing it in her mind to 4 + 10. What did she do? Does this strategy always work?
Efficiency	**Accuracy**
• What strategy did you use to solve 9 + 3? • How can you use 7 × 7 to solve 7 × 8? • Which facts do you "just know"? For which facts do you use a strategy?	• Crystal explains that 6 + 7 is 12. Is she correct? Explain how you know. • What is the answer to 7 × 8? How do you know it is correct (how might you check it)?
Creative writing ideas that address several components	
• Develop a "Face the facts" or "Ask Cougar" column (like Dear Abby) for the class. (Pick a fun name for the column that makes sense for the class, such as the school mascot.) Students send a letter about a tough fact. Rotate different students into the role of responder. The responder writes letters back, suggesting a strategy for the tough fact. • Create a strategy rhyme (e.g., If times four is giving me trouble, I'll remember to double and double). • Make a facts survival guide. Children prepare pages illustrating with visuals (e.g., ten frames or arrays) of how to find "tough" facts. • Write a yearbook entry to some facts (e.g., Dear 8 × 7, I …)	

(See McIntosh 1997 for many more ideas).

4. Quizzes

You may be surprised to see this section, given the major concerns raised earlier related to timed tests, but quizzes can be used effectively to assess efficiency as well as strategy use. Ensure that students "just know" their foundational facts before moving on to derived facts. Foundational facts are so named because they can be used to generate all the other facts using a strategy. Foundational facts in addition include one- and two-more-than, combinations that make ten, and doubles. For multiplication, they include × 1, × 2, × 5, and × 10. (See Kling and Bay-Williams, in press, for a discussion of foundational facts.) From these facts, we can derive all other facts. Quiz questions (see **fig. 3a**) can be used to see if students "just know" foundational facts.

Similarly, quizzes can be used to monitor facts that come more easily to students. For example, a quiz (see **fig. 3b)** assesses if students are recognizing the commutativity of addition for one-more-than facts. Notice that these examples are shorter, not timed, and also focus on strategies. The following adaptations can enhance the effectiveness of facts quizzes:

- Choose one of the problems above and write about how you solved it.
- Tell which helper fact you used the most on this quiz.
- Circle facts you "just knew." Highlight those for which you used a strategy.
- Circle facts you are sure about. Draw a square around facts that took you longer to solve.

Figure 3

Quizzes that focus on fluency are alternatives to timed tests.

(a) Quiz questions can be used to see if students "just know" foundational facts.

Solve these problems and tell how you solved them.

4 × 5 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

10 × 6 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

6 × 2 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

5 × 3 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

2 × 9 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

3 × 10 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

5 × 7 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

8 × 10 = ______ Check one: ☐ I used this strategy: ______________________
☐ I just knew.

(b) A quiz assesses if students recognize the commutativity of addition for one-more-than facts. Notice that these examples are shorter, not timed, and also focus on strategies. On completion, say to class, "Circle the row that was easier for you to solve. If they were both the same, write 'same'."

Solve these addition problems.

Row A	9 + 1 =	8 + 1	5 + 1 =	3 + 1 =	6 + 1
Row B	1 + 8 =	1 + 7 =	1 + 4	1 + 2	1 + 9 =

Meaningful fact assessment for teachers and students

We recognize that using timed tests is a deeply rooted practice for measuring basic fact mastery. We hope that we have effectively made a case for *why* this practice must change and *how* to make such a change. As the NCTM Assessment Principle states, "Assessment should support the learning of important mathematics and furnish useful information to both teachers and students" (NCTM 2000, p. 11). Using the range of assessments described above accomplishes these goals, as they provide an opportunity for meaningful, targeted feedback to students that far exceeds the "right or wrong, fast or slow" feedback provided by timed testing. In fact, these assessments infuse a fifth and critical category of assessment: self-assessment. Interviews, journals, *and* quizzes on basic facts can and should encourage students to reflect on which facts and strategies they know well and which ones are tough for them. This self-assessment can be effectively followed up by having children identify and record strategies that could be used to efficiently determine the "tough" facts in the future. Over time, this self-assessment practice encourages children to instinctively apply effective strategies for challenging facts they encounter. As both teachers and students critique their growth with use of appropriate strategies, efficiency, flexibility, and accuracy, then *true* fluency with basic facts can become a reality for every child.

Continued...

References

Baroody, Arthur J. 2006. "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them." *Teaching Children Mathematics* 13 (August): 22–31.

Bay-Williams, Jennifer M., and Gina Kling. In press. "Enriching Addition and Subtraction Fact Mastery through Games." *Teaching Children Mathematics.*

Boaler, Jo. 2012. "Timed Tests and the Development of Math Anxiety." Education Week. Online July 3, 2012.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Forbringer, Linda, and Allison J. Fahsl. 2010. "Differentiating Practice to Help Students Master Basic Facts." In *Responding to Diversity: Grades Pre-K–5*, edited by Dorothy Y. White and Julie S. Spitzer, pp. 7–22. Reston, VA: National Council of Teachers of Mathematics.

Henry, Valerie J., and Richard S. Brown. 2008. "First-Grade Basic Facts: An Investigation into Teaching and Learning of an Accelerated, High-Demand Memorization Standard." *Journal for Research in Mathematics Education* 39 (March): 153–83.

Hodges, Thomas E., Terry D. Rose, and April D. Hicks. 2012. "Interviews as RtI Tools." *Teaching Children Mathematics* 19 (August): 30–36.

Kamii, Constance, and Catherine Anderson. 2003. "Multiplication Games: How We Made and Used Them." *Teaching Children Mathematics* 10 (November): 135–41.

Kling, Gina. 2011. "Fluency with Basic Addition." *Teaching Children Mathematics* 18 (September): 80–88.

Kling, Gina, and Jennifer M. Bay-Williams. In press. "Three Steps to Mastering Multiplication Facts." *Teaching Children Mathematics.*

McIntosh, Margaret E. 1997. "500+ Writing Formats." *Mathematics Teaching in the Middle School* 2 (March/April): 354–58.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———. 2006. *Curriculum Focal Points for Prekindergarten through Grade 8: A Quest for Coherence*. Reston, VA: NCTM.

National Research Council (NRC). 2001. *Adding It Up: Helping Children Learn Mathematics.* Edited by Jeremy Kilpatrick, Jane Swafford, and Brad Findell. Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academies Press.

Ramirez, Gerardo, Elizabeth A. Gunderson, Susan C. Levine, and Sian L. Beilock. 2013. "Math Anxiety, Working Memory, and Math Achievement in Early Elementary School." *Journal of Cognition and Development* 14 (2): 187–202.

SBAC (Smarter Balanced Assessment Consortium). 2012. *Smarter Balanced Assessments*. http://www.smarterbalanced.org/smarter-balanced-assessments/#item

Van de Walle, John. A., Karen S. Karp, and Jennifer M. Bay-Williams. 2013. *Elementary and Middle School Mathematics: Teaching Developmentally. Professional Development Edition*. New York: Pearson.

Van de Walle, John A., Lou Ann H. Lovin, Karen S. Karp, and Jennifer M. Bay-Williams. 2014. *Teaching Student Centered Mathematics:* Grades K–2. 2nd edition. New York: Pearson.

Wiliam, Dylan. 2011. *Embedded Formative Assessment.* Bloomington, IN: Solution Tree Press.

Learning Algebra from Worked Examples

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Presenting examples of both correctly and incorrectly worked solutions is a practical classroom strategy that helps students counter misconceptions about algebra.

"Make sense of . . ."

"Reason quantitatively . . ."

"Critique the reasoning of others . . ."

These phrases will sound familiar to anyone who has internalized the next stage of standards-based mathematics education—the Common Core State Standards, currently being implemented in more than forty states. With its focus on both content and process standards, the Common Core State Standards Initiative (CCSSI 2010) seeks to deepen students' understanding in mathematics. Although the Common Core standards were written to address all grades, the goal of increasing comprehension is especially critical in first-year algebra, which functions as a gate-keeper course to higher mathematics and education (Star and Rittle-Johnson 2009).

Solving equations is a specifically pivotal unit in beginning algebra, one that is often confusing and challenging for many students. Many misconceptions that prohibit students from developing a deeper understanding of algebra have occurred and persisted since their early learning experiences. For example, misconceptions surrounding the generalizability of addition or the meaning of the equals sign that may begin in elementary school negatively affect students' understanding of variables and solving equations in beginning algebra classes (Samo 2009; Warren 2003). As a result of an increased focus on reasoning and sense making and less emphasis on procedural repetition (NCTM 2000) as well as the adoption of the Common Core State Standards (CCSSI 2010), algebra teachers may need additional tools to correct commonly held misconceptions. Most important, traditional instruction, even when designed to address misconceptions, often does not provide for sufficient conceptual change in student understanding (Vlassis 2004).

For students to be successful in algebra, they must have a truly conceptual understanding of key algebraic features as well as the procedural skills to complete a problem. One strategy to correct students' misconceptions combines the use of worked example problems in the classroom with student self-explanation. *Self-explanation* is the "activity of generating explanations to oneself" (Chi 2000, p. 164), especially "in attempt to make sense of new information" (p. 163) as one reads or studies.

A *worked example problem*, to be differentiated from *working an example problem*, shows students an already completed problem and directs their attention to certain steps of the task as the focus of questioning. Self-explanation, then, specifically encourages students to identify the reasoning behind the steps that they see carried out and to explain why these steps were completed. This strategy of providing worked example problems coupled with prompts for self-explanation has recently been shown to influence students' learning positively in both traditional (Booth, Koedinger, and Paré-Blagoev 2011) and computer-based classrooms (Booth et al. 2013). In addition, the Institute for Education Sciences promotes worked examples as a worthwhile instructional practice (Pashler et al. 2007), and several studies have shown that the technique has promise in classroom settings (Ward and Sweller 1990; Schwonke et al. 2009).

A unique and powerful aspect of using worked examples in the classroom occurs with the inclusion of examples of both correct and incorrect solutions (subsequently referred to as correct and incorrect examples). Some educators may fear that exposing students to incorrect examples may increase misconceptions, but research has shown otherwise (Booth et al. 2013). Using incorrect examples forces students to think about the steps that have been carried out and the reasons why these actions are wrong and then to confront their own possible underlying misconceptions. The desired result is a deeper understanding of mathematics for all students, regardless of prior skill level.

Linking to the Common Core

Solving equations is a key unit in beginning algebra, and the Common Core State Standards (2010) address this content area under the following two standards:

> A-REI.1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
>
> A-REI.2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. (p. 65)

With this phrasing, the Common Core Standards suggest that solving equations alone is not sufficient; students must demonstrate a deeper understanding of the concepts underlying the process of equation solving. When using the strategies suggested here, students must do more than simply view worked examples; rather, they should engage in reasoning and discussions about the examples. Self-explanation becomes an additional tool for teachers to encourage students to "make sense of problems," "reason abstractly and quantitatively," "construct viable arguments," and "critique the reasoning of others" (CCSSI 2010, pp. 6–7).

By using probing questions that require students to explain a previously worked example, teachers can ensure that students are making sense of what solving equations really entails. Students also engage in reasoning while constructing explanations and strengthen critical thinking skills while critiquing the correct or incorrect solutions. These tasks, carried out in conjunction with the Common Core State Standards, serve to promote deeper understanding of solving equations, which will help students of all ability levels prepare for higher-level mathematics.

Using examples in a computer-based activity

Correct and incorrect examples have been used successfully in a classroom setting through a computer program, Cognitive Tutor, that responds to students' ability by giving more or less practice according to their patterns of answering questions. Students receive immediate feedback about their choice of explanations. The answer turns green if the student responds correctly. If the student answers incorrectly, the answer turns red, and a pop-up box appears with a suggestion to review certain aspects of the problem. This feedback has been thought to increase the effectiveness of this type of exercise. In this computer-based classroom setting, worked example problems were incorporated by replacing a procedural problem in Cognitive Tutor.

One exercise using a correctly worked example to address a misconception began like this:

> Jose was asked to solve the following equation for *x*, and he made a *good* first step to solve the problem. Look at his first step:
>
> $4/x = -7$
>
> $4 = -7x$

Isolating this step, students might consider the significance of variables in the denominator and performing the same operation on both sides of the equation. Students were then prompted to answer two questions: "What did Jose do?" and "Why is that a *good* step for Jose to take?" The latter requires a two-part answer explaining both why the step was valid and why it was helpful. Students answered each of the three questions on the computer, using multiple drop-down boxes to construct complete answers. Given multiple options to choose from in each drop-down box, students had available a total of 180 complete sentences for the first question alone. Sample answers that students could construct are provided in **table 1**.

Students were explicitly told when a worked example was incorrect, as in this instance:

> Laura was asked to solve the following equation for *x*, and she made a *wrong* first step to solve the problem. Look at her first step:
>
> $2 = 5x - 3$
>
> $2 = 2x$

This type of example helps students address misconceptions about combining like terms. Again, two questions followed: "What did Laura do?" and "Why is that a *wrong* step for Laura to take?" In this case, the second question can be approached in one of two ways: "It is invalid because . . ." or "It is valid but not helpful because. . . ." Hundreds of sentence combinations could be created; a few sample answers for this question appear in **table 2**.

One benefit of using Cognitive Tutor is that it allows for a controlled research environment in an actual classroom. Results from a study of these types of example problems (Booth et al. 2013) show that students perform better overall on an equation-solving posttest after receiving both correctly worked and incorrectly worked examples (rather than receiving only correctly worked examples). The posttest included both procedural and conceptual measures of growth, and the study also found that students improved their conceptual knowledge the most after they had seen both the correct and incorrect examples. Contrary to the position that students with little prior knowledge should be shielded from incorrect examples, results showed that students who performed poorly on the pretest and then analyzed only the incorrect examples showed the most gain in conceptual understanding.

Table 1

Sample Constructed Answers about Jose's Worked Problem

What did Jose do?	**Why is that a *good* step for Jose to take?**	
	It is valid because…	***It is helpful because…***
He multiplied both sides by x.*	It combines like terms that are constant.	It put all the constant terms together.
He added x to both sides.	It combines like terms that are variable terms.	It isolated the variable.
He divided both sides by x.	It does the same operation to both sides.*	It moves the variables out of the denominator.*

*Correct answers

Table 2

Sample Constructed Answers about Laura's Worked Problem

What did Laura do?	**Why is that a *wrong* step for Laura to take?**	
	It is invalid because…	***It is valid but not helpful because…***
She added 3 to the right side.	It combines terms that are not like terms.*	It didn't reduce the number of terms.
She added –3 to 5x.*	It performs the operation to only one side of the equation.	It made the numbers more complicated.
She subtracted 3 from 5x.*	It performs different operations to both sides of the equation.	

*Correct answers

Using examples in a traditional classroom

Another way to incorporate this approach in a traditional classroom is through student homework. Even without the immediate feedback of a computer-based system, students displayed better conceptual understanding after completing assignments that included analyzing correctly worked and incorrectly worked examples. In one ongoing project, alternative versions of homework worksheets were created in which half the practice problems were replaced with worked examples for students to explain, and each example was followed by a corresponding problem that students would solve independently. A mix of clearly marked correct and incorrect solutions were provided and accompanied by prompts asking the students to explain something about the example. These prompts were designed to target specific common misconceptions and call students' attention to key conceptual features of the problem.

Continued…

Consider the example in **figure 1**. Students are asked to look at Eliza's solution and prompted to think about why Eliza's steps were appropriate. Then, at the right, students face a matched problem to solve on their own. In **figure 2**, students are presented with Jackson's flawed attempt to solve the problem and then asked questions about what was done incorrectly. Finally, students face a similar problem to solve independently. These problems continue to target common misconceptions by focusing student attention on the negative signs and combining like terms.

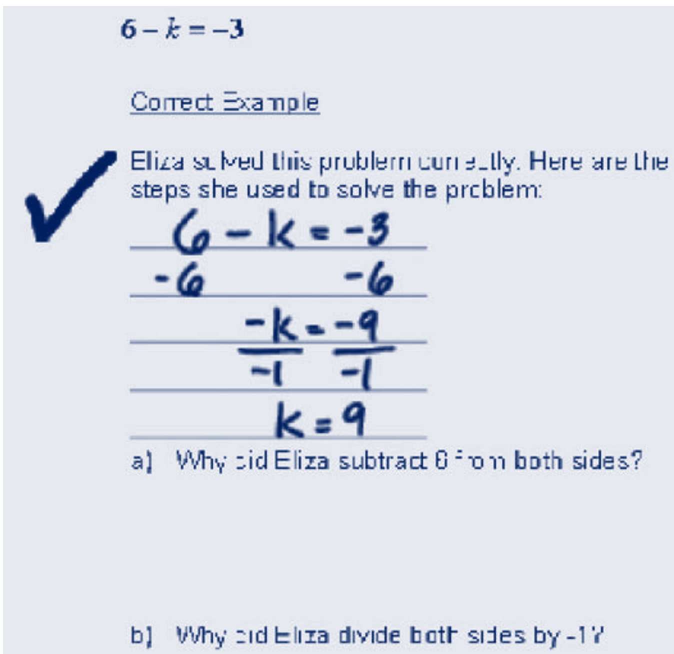
Student work is shown in **figure 3**.

This approach was not limited to use in solving equations. Worked examples and justification were also used to address misconceptions regarding word problems, problems with graphs, and number lines (see **fig. 4**).

Figure 1

Questions prompt students to analyze a correctly worked example.

Example



$6 - k = -3$

Correct Example

Eliza solved this problem correctly. Here are the steps she used to solve the problem:

$6 - k = -3$
$-6 \quad -6$
$-k = -9$
$\frac{-k}{-1} = \frac{-9}{-1}$
$k = 9$

a) Why did Eliza subtract 6 from both sides?

b) Why did Eliza divide both sides by -1?

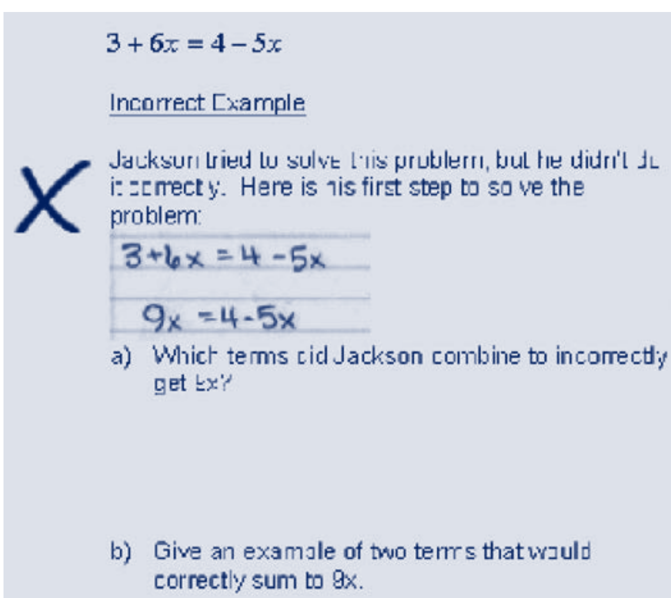
Matched Problem

$-6 - k = 3$

Figure 2

Questions prompt students to analyze an incorrectly worked example.

Example



$3 + 6x = 4 - 5x$

Incorrect Example

Jackson tried to solve this problem, but he didn't do it correctly. Here is his first step to solve the problem:

$3 + 6x = 4 - 5x$
$9x = 4 - 5x$

a) Which terms did Jackson combine to incorrectly get 9x?

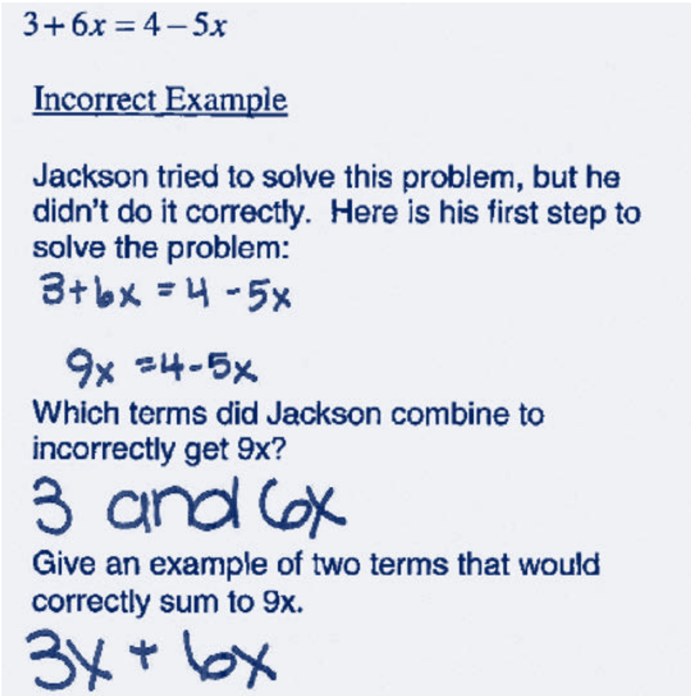
b) Give an example of two terms that would correctly sum to 9x.

Matched Problem

$-6x + 3 = 4 - 5x$

Figure 3

Students analyze mathematical reasoning.



$3+6x=4-5x$

Incorrect Example

Jackson tried to solve this problem, but he didn't do it correctly. Here is his first step to solve the problem:

3+6x = 4 - 5x

9x = 4-5x

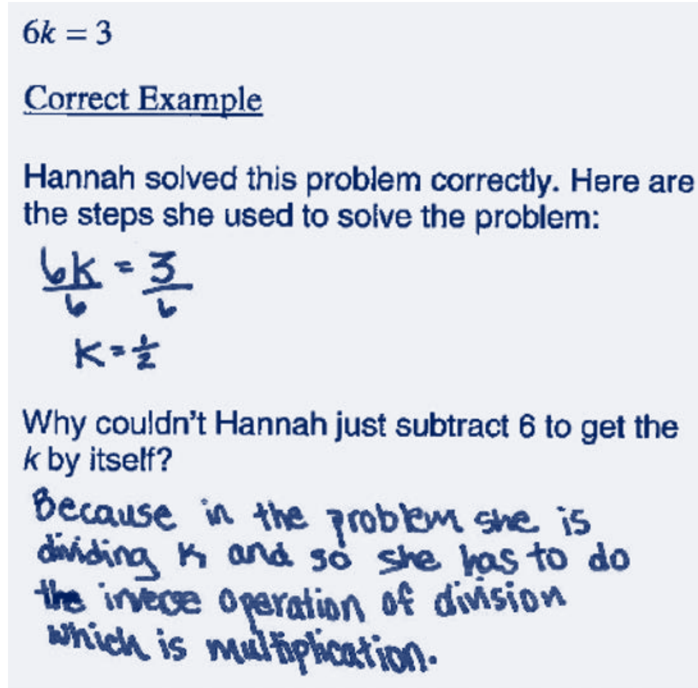
Which terms did Jackson combine to incorrectly get 9x?

3 and 6x

Give an example of two terms that would correctly sum to 9x.

3x + 6x

(a)



$6k=3$

Correct Example

Hannah solved this problem correctly. Here are the steps she used to solve the problem:

6k/6 = 3/6

k = 1/2

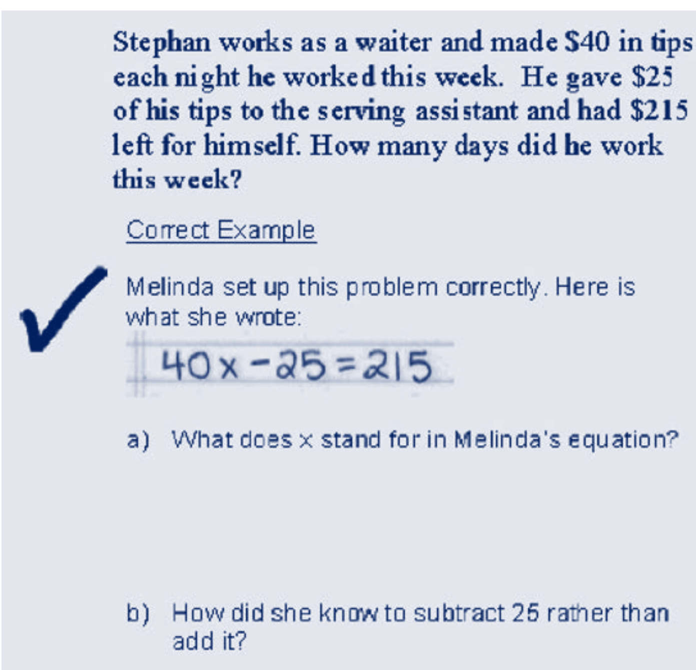
Why couldn't Hannah just subtract 6 to get the *k* by itself?

Because in the problem she is dividing k and so she has to do the invece operation of division which is multiplication.

(b)

Figure 4

Understanding word problems and graphs can also be improved through analyzing worked examples.



Stephan works as a waiter and made $40 in tips each night he worked this week. He gave $25 of his tips to the serving assistant and had $215 left for himself. How many days did he work this week?

Correct Example

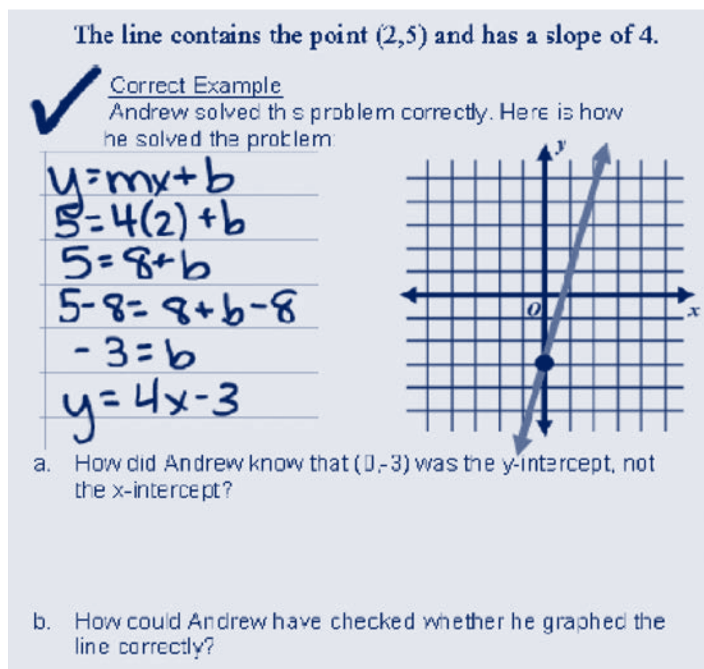
Melinda set up this problem correctly. Here is what she wrote:

40x - 25 = 215

a) What does x stand for in Melinda's equation?

b) How did she know to subtract 25 rather than add it?

(a)



The line contains the point (2,5) and has a slope of 4.

Correct Example

Andrew solved this problem correctly. Here is how he solved the problem:

y = mx + b

5 = 4(2) + b

5 = 8 + b

5 - 8 = 8 + b - 8

-3 = b

y = 4x - 3

a. How did Andrew know that (0,-3) was the y-intercept, not the x-intercept?

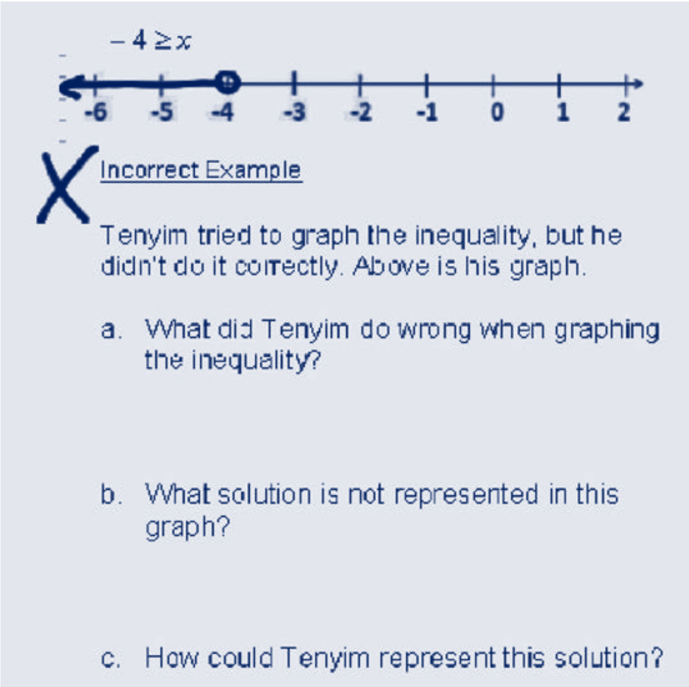
b. How could Andrew have checked whether he graphed the line correctly?

(b)

Continued...

Figure 4 continued

Understanding word problems and graphs can also be improved through analyzing worked examples.



(c)

The Student Experience

Teachers' initial hesitation to using both correct and incorrect examples may stem from concerns that students will have extra work or be confused by the incorrect examples. When students were first shown the example exercises, they often expressed confusion, saying things like "What is this . . . ?" or "I don't know what to do." However, as students moved on with additional worked examples, their comments changed. One student reflected, "I'm not really sure what she did . . ." before going on to answer the problem correctly. Another student exclaimed, "Oh, it makes me think!" During the exercises, students were actively critiquing the reasoning of others. For instance, in response to Laura's worked example, one student explained, "She subtracted, like, 5*x* minus 3*x*, but she shouldn't have."

When introducing a new strategy, teachers may face resistance from students. Such was the case with students who had completed units incorporating the worked example problems shown here. In the classrooms participating in the study detailed by Booth et al. (2013), students were first surprised by the new approach but then acknowledged that they were actually learning more. Sixty-two percent of students reported liking the module, although one student noted, "It was hard to do the 'find what's wrong' [problems]." However, 82 percent of students reported believing that the module was helping them learn. Sample responses from the student survey included these: "It made you do math"; "It taught me new ways to look at things"; "It helped me to learn things a little better"; and "Well, it helped when I got back to taking the posttest because I think I got more answers correct." In brief, although students found the process somewhat challenging, they reported that it helped them learn.

Targeting Student Misconceptions

Traditional mathematics instruction in solving equations often involves the teacher demonstrating the necessary steps to solve for the variable. Even when probing questions are asked or exploratory activities are completed, students may not be prompted to challenge their misconceptions explicitly. Incorporating worked example problems into the classroom offers a way for students to confront their misconceptions and to replace them with a deeper conceptual understanding. The most growth in student learning was found in a classroom environment in which students consider both correct and incorrect examples (Booth et al. 2013). In light of the Common Core push for reasoning skills, it is imperative that educators have access to strategies to promote conceptual understanding.

Results from the two research studies—Booth, Koedinger, and Paré-Blagoev (2011) and Booth et al. (2013)—suggest that for students to correct misconceptions, they need to see more worked examples, engage in discussion or explanation of these examples, and critique the reasoning behind incorrect mathematical statements. Worked examples here have been introduced through computer-based and homework-based methods. Teachers may be able to replicate these results by adapting the key components of this strategy—focused questions and student discussion on already completed examples—in a variety of ways.

Coupling both correctly and incorrectly worked examples with self-explanation allows students to confront their own misconceptions. Unlike other methods for remediating student difficulties, this type of intervention is beneficial for students at all ability levels, a key factor when working toward student achievement in a beginning algebra course. This approach affords all students the opportunity to explore mathematical reasoning and build a sound conceptual understanding of algebra.

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References

Booth, Julie L., Kenneth R. Koedinger, and Juliana E. Paré-Blagoev. 2011. "Testing the Worked Example Principle in Real-World Classrooms." Paper presented at the biannual meeting of the Society for Research in Child Development, Montreal.

Booth, Julie L., Karin E. Lange, Kenneth R. Koedinger, and Kristie J. Newton. 2013. "Using Example Problems to Improve Student Learning in Algebra: Differentiating between Correct and Incorrect Examples." *Learning and Instruction 25* (June): 24–34.

Chi, Michelene T. H. 2000. "Self Explaining Expository Texts: The Dual Processes of Generating Inferences and Repairing Mental Models." In *Advances in Instructional Psychology*, edited by Robert Glaser, pp. 161–238. Mahwah, NJ: Lawrence Erlbaum.

Common Core State Standards Initiative (CCSSI). 2010. *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/ assets/CCSSI_Math%20Standards.pdf

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Pashler, Harold, Patrice M. Bain, Brian A. Bottge, Arthur Graesser, Kenneth Koedinger, Mark McDaniel, and Janet Metcalfe. 2007. "Organizing Instruction and Study to Improve Student Learning" (NCER 2007-2004). Washington, DC: National Center for Education Research, Institute of Education Sciences, U.S. Department of Education.

Samo, Mashooque Ali. 2009. "Students' Perceptions about the Symbols, Letters, and Signs in Algebra and How Do These Affect Their Learning of Algebra: A Case Study in a Government Girls Secondary School Karachi." *International Journal for Mathematics Teaching and Learning* (January 5): 1–35.

Schwonke, Rolf, Alexander Renkl, Carmen Krieg, Jörg Wittwer, Vincent Aleven, and Ron Salden. 2009. "The Worked-Example Effect: Not an Artifact of Lousy Control Conditions." *Computers in Human Behavior 25* (2): 258–66.

Star, Jon R., and Bethany Rittle-Johnson. 2009. "Making Algebra Work: Instructional Strategies That Deepen Student Understanding, within and between Algebraic Representations." *ERS Spectrum 27*, 2 (March): 11–18.

Vlassis, Joelle. 2004. "Making Sense of the Minus Sign, or Becoming Flexible in 'Negativity.'" *Learning and Instruction 14* (5): 469–84.

Ward, Mark, and John Sweller. 1990. "Structuring Effective Worked Examples." *Cognition and Instruction 7* (1): 1–39.

Warren, Elizabeth. 2003. "The Role of Arithmetic Structure in the Transition from Arithmetic to Algebra." *Mathematics Education Research Journal 15* (2): 122–37.

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